

IDEALS OF POLYNOMIALS BETWEEN BANACH SPACES REVISITED

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ABSTRACT. Ideals of polynomials and multilinear operators between Banach spaces have been exhaustively investigated in the last decades. In this paper we introduce an unified (and more general) approach and propose some lines of investigation in this new framework. Among other results, we prove a Bohnenblust–Hille inequality in this more general setting.

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1. INTRODUCTION

Linear Functional Analysis emerged in the 30's after the publication of Banach's monograph. The investigation of polynomials and multilinear operators between normed spaces is, of course, the first natural step when moving from linear to nonlinear Functional Analysis. The theory of polynomials between normed spaces is a basic tool for the investigation of holomorphic mappings in Banach spaces. We recall that if E, F are normed spaces, a map $P : E \rightarrow F$ is called an m -homogeneous polynomial when there is an m -linear operator

$$A : E \times \cdots \times E \rightarrow F$$

such that

$$P(x) = A(x, \dots, x)$$

for all x in E . Continuity is defined as usual when dealing with metric spaces, and it is well known that P is continuous if and only if

$$\|P\| := \sup_{\|x\| \leq 1} \|P(x)\| < \infty.$$

The basics of the theory of polynomials and multilinear operators between Banach spaces can be found in the classical books [24, 29]. Polynomials and multilinear operators have been exhaustively investigated by quite different viewpoints. While polynomials are suitable to investigation of the holomorphic mappings, multilinear operators are commonly explored in the context of the extension of the operators ideals theory to the nonlinear setting. The notion of ideals of polynomials between Banach spaces is due to Pietsch [40]. The natural extension to multilinear operators and polynomials was designed by Pietsch some years later [39]. Nowadays, ideals of polynomials

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and multilinear operators are explored by several authors in different directions (see, for instance, [1, 4, 5, 7, 8, 9, 10, 11, 16, 26, 34]). In this paper we are mainly interested in the theory of ideals of polynomials and ideals of multilinear operators between Banach spaces. We propose an unified approach to the subject and some themes for future research.

2. IDEALS OF POLYNOMIALS AND MULTILINEAR OPERATORS: THE CLASSIC DEFINITIONS

We first recall the classical definition of operator ideals.

Definition 2.1 (Operator ideal). An *operator ideal* is a class \mathcal{I} of continuous linear operators between Banach spaces such that for all Banach space E and F , its components

$$\mathcal{I}(E; F) := \mathcal{L}(E; F) \cap \mathcal{I}$$

satisfy:

(Oa): $\mathcal{I}(E; F)$ is a linear subspace of $\mathcal{L}(E; F)$ which contains the finite rank operators;

(Ob): the ideal property: if $u \in \mathcal{I}(E; F)$, $v \in \mathcal{L}(G; E)$ and $t \in \mathcal{L}(F; H)$, then

$$t \circ u \circ v \in \mathcal{I}(G; H).$$

Moreover, \mathcal{I} is said to be a *(quasi-) normed operator ideal* if there exists a map $\|\cdot\|_{\mathcal{I}} : \mathcal{I} \rightarrow [0, \infty)$ satisfying:

(O1): $\|\cdot\|_{\mathcal{I}}$ restricted to $\mathcal{I}(E; F)$ is a (quasi-) norm, for all Banach spaces E, F ;

(O2): $\|id_{\mathbb{K}} : \mathbb{K} \rightarrow \mathbb{K} : id_{\mathbb{K}}(\lambda) = \lambda\|_{\mathcal{I}} = 1$;

(O3): if $u \in \mathcal{I}(E; F)$, $v \in \mathcal{L}(G; E)$ and $t \in \mathcal{L}(F; H)$, then

$$\|t \circ u \circ v\|_{\mathcal{I}} \leq \|t\| \|u\|_{\mathcal{I}} \|v\|.$$

When all the components $\mathcal{I}(E; F)$ are complete under the (quasi-) norm $\|\cdot\|_{\mathcal{I}}$ above, then \mathcal{I} is called a *(quasi-) Banach operator ideal*.

For the multilinear operators we have the following concepts:

Definition 2.2. Let E_1, \dots, E_m, F be normed spaces. A multilinear mapping $T \in \mathcal{L}(E_1, \dots, E_m; F)$ is said to be of *finite type* if there exist $k \in \mathbb{N}$, $\varphi_i^{(j)} \in E'_j$ and $b_i \in F$ for $i = 1, \dots, k$ and $j = 1, \dots, m$, such that

$$T(x_1, \dots, x_m) = \sum_{i=1}^k \varphi_i^{(1)}(x_1) \cdots \varphi_i^{(m)}(x_m) b_i.$$

We shall represent by $\mathcal{L}_f(E_1, \dots, E_m; F)$ the subspace of all finite type members of $\mathcal{L}(E_1, \dots, E_m; F)$.

The vector space $\mathcal{L}_f(E_1, \dots, E_m; F)$ satisfies the *ideal property*, that is, if $T \in \mathcal{L}_f(E_1, \dots, E_m; F)$, $u_j \in \mathcal{L}(G_j; E_j)$ for $j = 1, \dots, m$, and $t \in \mathcal{L}(F; H)$, then $t \circ T \circ (u_1, \dots, u_m) \in \mathcal{L}_f(G_1, \dots, G_m; H)$.

Definition 2.3 (Ideal of multilinear mappings). For each positive integer m , let \mathcal{L}_m denote the class of all continuous m -linear operators between Banach spaces. An *ideal of multilinear mappings* \mathcal{M} is a subclass of the class $\mathcal{L} = \bigcup_{m=1}^{\infty} \mathcal{L}_m$ of all continuous multilinear operators between Banach spaces such that for a positive integer m , Banach spaces E_1, \dots, E_m and F , the components

$$\mathcal{M}_m(E_1, \dots, E_m; F) := \mathcal{L}_m(E_1, \dots, E_m; F) \cap \mathcal{M}$$

satisfy:

(Ma): $\mathcal{M}_m(E_1, \dots, E_m; F)$ is a linear subspace of $\mathcal{L}_m(E_1, \dots, E_m; F)$ which contains the m -linear mappings of finite type;

(Mb): the ideal property: if $T \in \mathcal{M}_m(E_1, \dots, E_m; F)$, $u_j \in \mathcal{L}_1(G_j; E_j)$ for $j = 1, \dots, m$, and $t \in \mathcal{L}_1(F; H)$, then

$$t \circ T \circ (u_1, \dots, u_m) \in \mathcal{M}_m(G_1, \dots, G_m; F).$$

Moreover, \mathcal{M} is said to be a *(quasi-) normed ideal of multilinear mappings* if there exists a map $\|\cdot\|_{\mathcal{M}} : \mathcal{M} \rightarrow [0, \infty)$ satisfying:

(M1): $\|\cdot\|_{\mathcal{M}}$ restricted to $\mathcal{M}_m(E_1, \dots, E_m; F)$ is a (quasi-) norm, for all $m \in \mathbb{N}$ and Banach spaces E_1, \dots, E_m, F ;

(M2): $\|T_m : \mathbb{K}^m \rightarrow \mathbb{K} : T(\lambda_1, \dots, \lambda_m) = \lambda_1 \cdots \lambda_m\|_{\mathcal{M}} = 1$, for all $m \in \mathbb{N}$;

(M3): If $T \in \mathcal{M}_m(E_1, \dots, E_m; F)$, $u_j \in \mathcal{L}_1(G_j; E_j)$ for $j = 1, \dots, m$, and $t \in \mathcal{L}_1(F; H)$, then

$$\|t \circ T \circ (u_1, \dots, u_m)\|_{\mathcal{M}} \leq \|t\| \|T\|_{\mathcal{M}} \|u_1\| \cdots \|u_m\|.$$

When all the components $\mathcal{M}_m(E_1, \dots, E_m; F)$ are complete under the (quasi-) norm $\|\cdot\|_{\mathcal{M}}$ above, \mathcal{M} is said to be a *(quasi-) Banach ideal of multilinear mappings*. For a fixed ideal of multilinear mappings \mathcal{M} and a positive integer $m \in \mathbb{N}$, the class

$$\mathcal{M}_m := \bigcup_{E_1, \dots, E_m, F} \mathcal{M}_m(E_1, \dots, E_m; F)$$

is called an *ideal of m -linear mappings*.

For the homogeneous polynomials we have the following concepts:

Definition 2.4. Let E, F be normed spaces. A polynomial $P \in \mathcal{P}(^m E; F)$ is said to be of *finite type* if there exists $k \in \mathbb{N}$, $\varphi_i \in E'$ and $b_i \in F$ for $i = 1, \dots, k$, such that

$$P(x) = \sum_{i=1}^k \varphi_i(x)^m b_i.$$

We shall represent by $\mathcal{P}_f(^m E; F)$ the subspace of all finite-type members of $\mathcal{P}(^m E; F)$.

The vector space $\mathcal{P}_f(^m E; F)$ satisfies the *ideal property*, that is, if $P \in \mathcal{P}_f(^m E; F)$, $u \in \mathcal{L}(G; E)$, and $t \in \mathcal{L}(F; H)$, then $t \circ P \circ u \in \mathcal{P}_f(^m G; H)$.

Definition 2.5 (Polynomial ideal). For each positive integers m , let \mathcal{P}_m denote the class of all continuous m -homogeneous polynomials between Banach spaces. A *polynomial ideal* \mathcal{Q} (or *ideal of homogeneous polynomials*) is a subclass of the class $\mathcal{P} = \bigcup_{m=1}^{\infty} \mathcal{P}_m$ of all continuous homogeneous polynomials between Banach spaces such that for all $m \in \mathbb{N}$ and all Banach spaces E and F , the components

$$\mathcal{Q}_m(^m E; F) := \mathcal{P}_m(^m E; F) \cap \mathcal{Q}$$

satisfy:

(Pa): $\mathcal{Q}_m(^m E; F)$ is a linear subspace of $\mathcal{P}_m(^m E; F)$ which contains the finity-type m -homogeneous polynomials;

(Pb): the ideal property: if $u \in \mathcal{L}_1(G; E)$ $P \in \mathcal{Q}_m(^m E; F)$ and $t \in \mathcal{L}_1(F; H)$, then

$$t \circ P \circ u \in \mathcal{Q}_m(^m G; H).$$

Moreover, \mathcal{Q} is said to be a *(quasi-) normed polynomial ideal* if there exists a map $\|\cdot\|_{\mathcal{Q}} : \mathcal{Q} \rightarrow [0, \infty)$ satisfying:

(P1): $\|\cdot\|_{\mathcal{Q}}$ restricted to $\mathcal{Q}_m(^m E; F)$ is a (quasi-) norm, for all $m \in \mathbb{N}$ and all Banach spaces E and F ;

(P2): $\|P_m : \mathbb{K} \rightarrow \mathbb{K} : P_m(\lambda) = \lambda^m\|_{\mathcal{Q}} = 1$, for all $m \in \mathbb{N}$;

(P3): If $u \in \mathcal{L}_1(G; E)$, $P \in \mathcal{Q}_m(^m E; F)$ and $t \in \mathcal{L}_1(F; H)$, then

$$\|t \circ P \circ u\|_{\mathcal{Q}} \leq \|t\| \|P\|_{\mathcal{Q}} \|u\|^m.$$

When all the components $\mathcal{Q}_m(^m E; F)$ are complete under the (quasi-) norm $\|\cdot\|_{\mathcal{Q}}$ above, then \mathcal{Q} is called a *(quasi-) Banach polynomial ideal*. For a fixed polynomial ideal \mathcal{Q} and a positive integer $m \in \mathbb{N}$, the class

$$\mathcal{Q}_m := \bigcup_{E, F} \mathcal{Q}_m(^m E; F)$$

is called an *ideal of m -homogeneous polynomials*.

3. BASIC RESULTS

A fact apparently overlooked in the literature is that every m -linear operator is in fact a polynomial (we thank Prof. Pilar Rueda and R. Aron for important conversations about it). More precisely, if

$$T : E_1 \times \cdots \times E_m \rightarrow F$$

is an m -linear operator then, denoting $E := E_1 \times \cdots \times E_m$, the map

$$\begin{aligned} P & : E \rightarrow F \\ P(x_1, \dots, x_m) & = T(x_1, \dots, x_m) \end{aligned}$$

is an m -homogeneous polynomial. This fact can be easily proved by using tensor products. So, one can wonder why to define separately ideals of polynomials and ideals of multilinear operators, having in mind that every m -linear operator is in fact an m -homogeneous polynomial. Well, we can give a couple of reasons for that. A more obvious reason is that when considering a multilinear operator as a polynomial we have

$$\|T(x_1, \dots, x_m)\| \leq \|T\| \|(x_1, \dots, x_m)\|,$$

and this estimate is less precise than

$$(1) \quad \|T(x_1, \dots, x_m)\| \leq \|T\| \|x_1\| \cdots \|x_m\|.$$

So, one cannot unify the theory of polynomial ideals and multilinear ideals just by realizing that multilinear operators are in fact homogeneous polynomials. In this section we unify the theory of polynomials and multilinear operators in a more careful analytic viewpoint. More precisely, if m is a given positive integer and n_1, \dots, n_m are positive integers such that $n_1 + \cdots + n_j = m$ we introduce the notion of (n_1, \dots, n_j) -homogeneous polynomial for $j \in \{1, \dots, m\}$. When $j = 1$ we have an m -homogeneous polynomial and when $j = m$ then we have an m -linear operator. This kind of maps will be called multipolynomials.

Definition 3.1. Let $m \in \mathbb{N}$ and $(n_1, \dots, n_m) \in \mathbb{N}^m$. A mapping $P : E_1 \times \cdots \times E_m \rightarrow F$ is said to be an (n_1, \dots, n_m) -homogeneous polynomial if, for each $i = 1, \dots, m$, the mapping $P(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_m) : E_i \rightarrow F$ is an n_i -homogeneous polynomial for every $x_1 \in E_1, \dots, x_{i-1} \in E_{i-1}, x_{i+1} \in E_{i+1}, \dots, x_m \in E_m$ fixed.

We shall denote by $\mathcal{P}_a({}^{n_1}E_1, \dots, {}^{n_m}E_m; F)$ the vector space of all (n_1, \dots, n_m) -homogeneous polynomials from the cartesian product $E_1 \times \cdots \times E_m$ into F . We shall represent by $\mathcal{P}({}^{n_1}E_1, \dots, {}^{n_m}E_m; F)$ the subspace of all continuous members of $\mathcal{P}_a({}^{n_1}E_1, \dots, {}^{n_m}E_m; F)$. For each $P \in \mathcal{P}_a({}^{n_1}E_1, \dots, {}^{n_m}E_m; F)$ we shall set

$$\|P\| = \sup \left\{ \|P(x_1, \dots, x_m)\| ; x_i \in E_i, \max_i \|x_i\|_{E_i} \leq 1 \right\}.$$

Our first step is to present a comprehensive list conditions which characterize continuous multipolynomials similarly as in (1).

Henceforth $\mathcal{L}_a^s({}^mE; F)$ denotes the space of all m -linear forms from $E \times \cdots \times E$ to F which are symmetric and for each $A \in \mathcal{L}_a({}^mE; F)$ the m -homogeneous polynomial $\hat{A} \in \mathcal{P}_a({}^mE; F)$ is defined by $\hat{A}(x) = Ax^m$ for every $x \in E$.

We recall some results from the theory of homogeneous polynomials between Banach spaces that will be useful in this paper (these results can be found, for instance, in [29, Theorem 1.10], [29, Theorem 2.2] and [29, Corollary 2.3])

- (Polarization Formula) If $A \in \mathcal{L}_a^s({}^mE; F)$, then for $x_0, \dots, x_m \in E$ we have

$$A(x_1, \dots, x_m) = \frac{1}{m!2^m} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_m A(x_0 + \varepsilon_1 x_1 + \cdots + \varepsilon_m x_m)^m.$$

- The mapping $A \mapsto \hat{A}$ induces a vector space isomorphism between $\mathcal{L}_a^s({}^mE; F)$ and $\mathcal{P}_a({}^mE; F)$.

- We have the inequalities

$$(2) \quad \|\hat{A}\| \leq \|A\| \leq \frac{m^m}{m!} \|\hat{A}\|,$$

for every $A \in \mathcal{L}_a^s({}^m E; F)$.

- A polynomial $P \in \mathcal{P}_a({}^m E; F)$ is continuous if and only if $\|P\| < \infty$.
- $\mathcal{P}({}^m E; F)$ is a Banach space under the norm $P \mapsto \|P\|$.
- The mapping $A \mapsto \hat{A}$ induces a topological isomorphism between $\mathcal{L}^s({}^m E; F)$ and $\mathcal{P}({}^m E; F)$.

We begin with a useful lemma:

Lemma 3.2. *Let E_1, \dots, E_m, F be normed spaces and $P \in \mathcal{P}_a({}^{n_1} E_1, \dots, {}^{n_m} E_m; F)$. If P is bounded by C on an open ball $B_{E_1 \times \dots \times E_m}((a_1, \dots, a_m); r)$ then P is bounded by $C \frac{n_1^{n_1}}{n_1!} \dots \frac{n_m^{n_m}}{n_m!}$ on the ball $B_{E_1 \times \dots \times E_m}((0, \dots, 0); r)$.*

Proof. Let $(x_1, \dots, x_m) \in B_{E_1 \times \dots \times E_m}((0, \dots, 0); r)$. We prove this by induction on m . When $m = 1$ it is just the already well-known result from the theory of homogeneous polynomials be-

tween Banach spaces (see [29, Lemma 2.5.]). If $m > 1$ then the multipolynomial $P \left(\underbrace{\cdot, \dots, \cdot}_{m-1}, y \right) \in \mathcal{P}_a({}^{n_1} E_1, \dots, {}^{n_{m-1}} E_{m-1}; F)$ is bounded by C on the ball $B_{E_1 \times \dots \times E_{m-1}}((a_1, \dots, a_{m-1}); r)$, for all $y \in B_{E_m}(a_m; r)$. The induction hypothesis implies that $P \left(\underbrace{\cdot, \dots, \cdot}_{m-1}, y \right)$ is bounded by $C \frac{n_1^{n_1}}{n_1!} \dots \frac{n_{m-1}^{n_{m-1}}}{n_{m-1}!}$ on the ball $B_{E_1 \times \dots \times E_{m-1}}((0, \dots, 0); r)$, whenever $y \in B_{E_m}(a_m; r)$.

We also have from [29, Theorem 2.2] that the n_m -linear mapping, denoted by $A(x_1, \dots, x_{m-1}, \cdot)$, associated to the polynomial $P(x_1, \dots, x_{m-1}, \cdot) \in \mathcal{P}_a({}^{n_m} E_m; F)$ can be taken symmetrical. Now applying the Polarization Formula [29, Theorem 1.10] to $A(x_1, \dots, x_{m-1})$ with $x_0 = a_m$ and $x_1 = \dots = x_{n_m} = \frac{x_m}{n_m}$ we get

$$\begin{aligned} & \|P(x_1, \dots, x_m)\| \\ &= n_m^{n_m} \left\| A(x_1, \dots, x_{m-1}) \left(\frac{x_m}{n_m} \right)^{n_m} \right\| \\ &= n_m^{n_m} \left\| \frac{1}{n_m! 2^{n_m}} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \dots \varepsilon_{n_m} A(x_1, \dots, x_{m-1}) \left(a_m + (\varepsilon_1 + \dots + \varepsilon_{n_m}) \frac{x_m}{n_m} \right)^{n_m} \right\| \\ &\leq \frac{n_m^{n_m}}{n_m! 2^{n_m}} \sum_{\varepsilon_j = \pm 1} \left\| A(x_1, \dots, x_{m-1}) \left(a_m + (\varepsilon_1 + \dots + \varepsilon_{n_m}) \frac{x_m}{n_m} \right)^{n_m} \right\| \\ &= \frac{n_m^{n_m}}{n_m! 2^{n_m}} \sum_{\varepsilon_j = \pm 1} \left\| P \left(\underbrace{\cdot, \dots, \cdot}_{m-1}, a_m + (\varepsilon_1 + \dots + \varepsilon_{n_m}) \frac{x_m}{n_m} \right) (x_1, \dots, x_{m-1}) \right\| \\ &\leq \frac{n_m^{n_m}}{n_m! 2^{n_m}} 2^{n_m} C \frac{n_1^{n_1}}{n_1!} \dots \frac{n_{m-1}^{n_{m-1}}}{n_{m-1}!} \\ &= C \frac{n_1^{n_1}}{n_1!} \dots \frac{n_{m-1}^{n_{m-1}}}{n_{m-1}!} \frac{n_m^{n_m}}{n_m!}. \end{aligned}$$

Then it follows that P is bounded by $C \frac{n_1^{n_1}}{n_1!} \dots \frac{n_m^{n_m}}{n_m!}$ on the ball $B_{E_1 \times \dots \times E_m}((0, \dots, 0); r)$, and the proof is complete. \square

Now we are ready to characterize continuous multipolynomials.

Theorem 3.3. *Let E_1, \dots, E_m, F be normed spaces and $P \in \mathcal{P}_a({}^{n_1} E_1, \dots, {}^{n_m} E_m; F)$. The following conditions are equivalent:*

- (i): P is continuous;

- (ii): P is continuous at the origin;
 (iii): There exists a constant $C \geq 0$ such that

$$\|P(x_1, \dots, x_m)\| \leq C \|x_1\|^{n_1} \cdots \|x_m\|^{n_m},$$

for all $(x_1, \dots, x_m) \in E_1 \times \cdots \times E_m$;

- (iv): $\|P\| < \infty$;
 (v): P is uniformly continuous on bounded subsets of $E_1 \times \cdots \times E_m$;
 (vi): P is bounded on every ball with finite radius;
 (vii): P is bounded on some ball;
 (viii): P is bounded on some ball with center at the origin.

Proof. The implications (i) \Rightarrow (ii) and (vi) \Rightarrow (vii) are obvious.

(ii) \Rightarrow (iii): Suppose P continuous at the origin. Then, there exist $\delta > 0$ such that

$$(x_1, \dots, x_m) \in E_1 \times \cdots \times E_m, \|(x_1, \dots, x_m)\| < \delta \Rightarrow \|P(x_1, \dots, x_m)\| < 1.$$

The inequality in (iii) is obvious if $x_i = 0$ for some $i = 1, \dots, m$. So we can assume $x_i \neq 0$ for all $i = 1, \dots, m$. Then,

$$\left\| \left(\frac{\delta x_1}{2 \|x_1\|}, \dots, \frac{\delta x_m}{2 \|x_m\|} \right) \right\| = \frac{\delta}{2} < \delta$$

and thus

$$\begin{aligned} \|P(x_1, \dots, x_m)\| &= \left(\frac{2}{\delta} \right)^{n_1 + \cdots + n_m} \|x_1\|^{n_1} \cdots \|x_m\|^{n_m} \left\| P \left(\frac{\delta x_1}{2 \|x_1\|}, \dots, \frac{\delta x_m}{2 \|x_m\|} \right) \right\| \\ &< \left(\frac{2}{\delta} \right)^{n_1 + \cdots + n_m} \|x_1\|^{n_1} \cdots \|x_m\|^{n_m}. \end{aligned}$$

This give us (iii) with $C = \left(\frac{2}{\delta} \right)^{n_1 + \cdots + n_m}$.

(iii) \Rightarrow (iv): If (iii) is true then we have in particular,

$$\|P(x_1, \dots, x_m)\| \leq C \|x_1\|^{n_1} \cdots \|x_m\|^{n_m} \leq C,$$

for all $x_1 \in E_1, \dots, x_m \in E_m$, with $\|x_1\|, \dots, \|x_m\| \leq 1$. This shows that $\|P\| \leq C$.

(iv) \Rightarrow (v): Let $a = (a_1, \dots, a_m)$, $x = (x_1, \dots, x_m) \in E_1 \times \cdots \times E_m$ with

$$\max_i \|x_i\| \leq r$$

and

$$\max_i \|a_i\| \leq r.$$

Then

$$\begin{aligned} (3) \quad \|P(a_1, \dots, a_{i-1}, \cdot, x_{i+1}, \dots, x_m)\| &\leq \|a_1\|^{n_1} \cdots \|a_{i-1}\|^{n_{i-1}} \|x_{i+1}\|^{n_{i+1}} \cdots \|x_m\|^{n_m} \|P\| \\ &\leq r^{n_1 + \cdots + n_{i-1} + n_{i+1} + \cdots + n_m} \|P\|. \end{aligned}$$

Let $A(a_1, \dots, a_{i-1}, x_{i+1}, \dots, x_m)$ be the symmetric n_i -linear form associated to $P(a_1, \dots, a_{i-1}, \cdot, x_{i+1}, \dots, x_m)$. From (2) we get

$$\begin{aligned} (4) \quad \|A(a_1, \dots, a_{i-1}, x_{i+1}, \dots, x_m)\| &\leq \frac{n_i^{n_i}}{n_i!} \|P(a_1, \dots, a_{i-1}, \cdot, x_{i+1}, \dots, x_m)\| \\ &\leq \frac{n_i^{n_i}}{n_i!} r^{n_1 + \cdots + n_{i-1} + n_{i+1} + \cdots + n_m} \|P\|, \end{aligned}$$

for every $i = 1, \dots, m$. Since $\|P\| < \infty$ the inequalities (3) and (4) show us that both the i -th ordinary polynomial $P(a_1, \dots, a_{i-1}, \cdot, x_{i+1}, \dots, x_m) \in \mathcal{P}_a({}^{n_i}E; F)$ as its associated multilinear

mapping $A(a_1, \dots, a_{i-1}, x_{i+1}, \dots, x_m) \in \mathcal{L}_a({}^{n_i}E; F)$ are continuous for each $i = 1, \dots, m$. Now we can write

$$\begin{aligned}
& \|P(x) - P(a)\| \\
& \leq \sum_{i=1}^m \|P(a_1, \dots, a_{i-1}, x_i, \dots, x_m) - P(a_1, \dots, a_i, x_{i+1}, \dots, x_m)\| \\
& = \sum_{i=1}^m \|A(a_1, \dots, a_{i-1}, x_{i+1}, \dots, x_m)(x_i)^{n_i} - A(a_1, \dots, a_{i-1}, x_{i+1}, \dots, x_m)(a_i)^{n_i}\| \\
& \leq \sum_{i=1}^m \left(\left\| \begin{aligned} & A(a_1, \dots, a_{i-1}, x_{i+1}, \dots, x_m)(x_i - a_i, x_i, \dots, x_i) + \dots \\ & \dots + A(a_1, \dots, a_{i-1}, x_{i+1}, \dots, x_m)(a_i, \dots, a_i, x_i - a_i) \end{aligned} \right\| \right) \\
& \leq \sum_{i=1}^m n_i \|A(a_1, \dots, a_{i-1}, x_{i+1}, \dots, x_m)\| \|x_i - a_i\| r^{n_i-1} \\
& \leq \sum_{i=1}^m n_i \left(\frac{n_i^{n_i}}{n_i!} r^{n_1 + \dots + n_{i-1} + n_{i+1} + \dots + n_m} \|P\| \right) \|x - a\| r^{n_i-1} \\
& \leq \left(\sum_{i=1}^m \frac{n_i^{n_i+1}}{n_i!} \right) r^{n_1 + \dots + n_m-1} \|P\| \|x - a\|,
\end{aligned}$$

and the uniform continuity of P on bounded subsets of $E_1 \times \dots \times E_m$ follows.

(v) \Rightarrow (i): Let us show that P is continuous at an arbitrary point $a = (a_1, \dots, a_m) \in E = E_1 \times \dots \times E_m$. Given $\varepsilon > 0$ it follows from the uniform continuity of P on bounded subsets that there exist $\delta_0 > 0$ such that, for every $x = (x_1, \dots, x_m), y = (y_1, \dots, y_m) \in B_E(0; \|a\| + 1)$,

$$\|x - y\| < \delta_0 \Rightarrow \|P(x) - P(y)\| < \varepsilon.$$

Defining $\delta = \min\{\delta_0, 1\}$ we get

$$x \in E_1 \times \dots \times E_m, \|x - a\| < \delta \Rightarrow \|P(x) - P(a)\| < \varepsilon$$

and thus P is continuous at the point a .

(iii) \Rightarrow (vi): Let B be a ball with center at $(a_1, \dots, a_m) \in E_1 \times \dots \times E_m$ and radius $r > 0$. For every $(x_1, \dots, x_m) \in B$ the hypothesis (iii) give us a constant $C \geq 0$ such that

$$\|P(x_1, \dots, x_m)\| \leq C \|x_1\|^{n_1} \dots \|x_m\|^{n_m} \leq C(r + \|a_1\|)^{n_1} \dots (r + \|a_m\|)^{n_m}$$

and so P is bounded on B .

(vii) \Rightarrow (viii): It follows immediately from the Lemma 3.2.

(viii) \Rightarrow (iv): Suppose that there exist $r > 0$ and $C \geq 0$ such that

$$\|P(x_1, \dots, x_m)\| \leq C, \forall (x_1, \dots, x_m) \in B_{E_1 \times \dots \times E_m}((0, \dots, 0); r).$$

Thus, given $x_1 \in E_1, \dots, x_m \in E_m$, with $\|x_1\|, \dots, \|x_m\| \leq 1$, we have $(\frac{r}{2}x_1, \dots, \frac{r}{2}x_m) \in B_{E_1 \times \dots \times E_m}((0, \dots, 0); r)$ and hence

$$\|P(x_1, \dots, x_m)\| = \left(\frac{2}{r}\right)^{n_1 + \dots + n_m} \left\|P\left(\frac{r}{2}x_1, \dots, \frac{r}{2}x_m\right)\right\| \leq C \left(\frac{2}{r}\right)^{n_1 + \dots + n_m}.$$

□

The previous results can be combined to prove that the mapping $P \mapsto \|P\|$ defines a norm on the vector space $\mathcal{P}({}^{n_1}E_1, \dots, {}^{n_m}E_m; F)$ and, moreover, $\mathcal{P}({}^{n_1}E_1, \dots, {}^{n_m}E_m; F)$ is complete. The proof of the next results are standard and we omit.

Lemma 3.4. *Let E_1, \dots, E_m, F be normed spaces and let (P_j) be a sequence in $\mathcal{P}_a({}^{n_1}E_1, \dots, {}^{n_m}E_m; F)$ such that the limit $P(x_1, \dots, x_m) = \lim_{j \rightarrow \infty} P_j(x_1, \dots, x_m)$ exists for every $(x_1, \dots, x_m) \in E_1 \times \dots \times E_m$. Then $P \in \mathcal{P}_a({}^{n_1}E_1, \dots, {}^{n_m}E_m; F)$.*

Theorem 3.5. *Let E_1, \dots, E_m be normed spaces and let F be a Banach space. Then $\mathcal{P}(^{n_1}E_1, \dots, ^{n_m}E_m; F)$ is a Banach space under the norm $P \mapsto \|P\|$.*

4. THE UNIFIED APPROACH: MULTIPOLYNOMIAL IDEALS

In this section we present a more general and unifying polynomial and multilinear operator ideals version. We shall recall that our result reaches, as the extreme cases, the operator ideals notion for polynomials and for multilinear mappings.

Definition 4.1. Let E_1, \dots, E_m, F be normed spaces. A multipolynomial $P \in \mathcal{P}(^{n_1}E_1, \dots, ^{n_m}E_m; F)$ is said to be of *finite type* if there exists $k \in \mathbb{N}$, $\varphi_i^{(j)} \in E_j'$ and $b_i \in F$ with $i = 1, \dots, k$ and $j = 1, \dots, m$, such that

$$P(x_1, \dots, x_m) = \sum_{i=1}^k \varphi_i^{(1)}(x_1)^{n_1} \cdots \varphi_i^{(m)}(x_m)^{n_m} b_i.$$

We shall represent by $\mathcal{P}_f(^{n_1}E_1, \dots, ^{n_m}E_m; F)$ the subspace of all finite type members of $\mathcal{P}(^{n_1}E_1, \dots, ^{n_m}E_m; F)$.

Note that the vector space $\mathcal{P}_f(^{n_1}E_1, \dots, ^{n_m}E_m; F)$ satisfies the *ideal property*, that is, if $P \in \mathcal{P}_f(^{n_1}E_1, \dots, ^{n_m}E_m; F)$, $u_j \in \mathcal{L}(G_j; E_j)$ for $j = 1, \dots, m$, and $t \in \mathcal{L}(F; H)$, then $t \circ P \circ (u_1, \dots, u_m) \in \mathcal{P}_f(^{n_1}G_1, \dots, ^{n_m}G_m; H)$.

Definition 4.2 (Multipolynomial ideal). For each $m \in \mathbb{N}$ and multi-index $(n_1, \dots, n_m) \in \mathbb{N}^m$, let $\mathcal{P}_m^{(n_1, \dots, n_m)}$ denote the class of all continuous (n_1, \dots, n_m) -homogeneous polynomials between Banach

spaces. A *multipolynomial ideal* \mathcal{U} is a subclass of the class $\mathcal{V} = \bigcup_{m=1}^{\infty} \left(\bigcup_{(n_1, \dots, n_m) \in \mathbb{N}^m} \mathcal{P}_m^{(n_1, \dots, n_m)} \right)$ of all continuous multipolynomials between Banach spaces such that for all $m \in \mathbb{N}$, multi-index $(n_1, \dots, n_m) \in \mathbb{N}^m$ and all Banach spaces E_1, \dots, E_m and F , the components

$$\mathcal{U}_m^{(n_1, \dots, n_m)}(^{n_1}E_1, \dots, ^{n_m}E_m; F) := \mathcal{P}(^{n_1}E_1, \dots, ^{n_m}E_m; F) \cap \mathcal{U}$$

satisfy:

- (Ua): $\mathcal{U}_m^{(n_1, \dots, n_m)}(^{n_1}E_1, \dots, ^{n_m}E_m; F)$ is a linear subspace of $\mathcal{P}(^{n_1}E_1, \dots, ^{n_m}E_m; F)$ which contains the (n_1, \dots, n_m) -homogeneous polynomials of finite type ;
- (Ub): the ideal property: if $P \in \mathcal{U}_m^{(n_1, \dots, n_m)}(^{n_1}E_1, \dots, ^{n_m}E_m; F)$, $u_j \in \mathcal{L}_1(G_j; E_j)$ for $j = 1, \dots, m$, and $t \in \mathcal{L}_1(F; H)$, then

$$t \circ P \circ (u_1, \dots, u_m) \in \mathcal{U}_m^{(n_1, \dots, n_m)}(^{n_1}G_1, \dots, ^{n_m}G_m; F).$$

Moreover, \mathcal{U} is said to be a *(quasi-) normed multipolynomial ideal* if there exists a map $\|\cdot\|_{\mathcal{U}} : \mathcal{U} \rightarrow [0, \infty)$ satisfying:

- (U1): $\|\cdot\|_{\mathcal{U}}$ restricted to $\mathcal{U}_m^{(n_1, \dots, n_m)}(^{n_1}E_1, \dots, ^{n_m}E_m; F)$ is a (quasi-) norm, for all $m \in \mathbb{N}$, multi-index $(n_1, \dots, n_m) \in \mathbb{N}^m$ and all Banach spaces E_1, \dots, E_m, F ;
- (U2): $\left\| P_m^{(n_1, \dots, n_m)} : \mathbb{K}^m \rightarrow \mathbb{K} : P_m^{(n_1, \dots, n_m)}(\lambda_1, \dots, \lambda_m) = \lambda_1^{n_1} \cdots \lambda_m^{n_m} \right\|_{\mathcal{U}} = 1$, for all $m \in \mathbb{N}$ and $(n_1, \dots, n_m) \in \mathbb{N}^m$;
- (U3): If $P \in \mathcal{U}_m^{(n_1, \dots, n_m)}(^{n_1}E_1, \dots, ^{n_m}E_m; F)$, $u_j \in \mathcal{L}_1(G_j; E_j)$ for $j = 1, \dots, m$, and $t \in \mathcal{L}_1(F; H)$, then

$$\|t \circ P \circ (u_1, \dots, u_m)\|_{\mathcal{U}} \leq \|t\| \|P\|_{\mathcal{U}} \|u_1\|^{n_1} \cdots \|u_m\|^{n_m}.$$

When all the components $\mathcal{U}_m^{(n_1, \dots, n_m)}(^{n_1}E_1, \dots, ^{n_m}E_m; F)$ are complete under the (quasi-) norm $\|\cdot\|_{\mathcal{U}}$ above, then \mathcal{U} is called a *(quasi-) Banach multipolynomial ideal*. For a fixed multipolynomial ideal \mathcal{U} , a positive integer $m \in \mathbb{N}$ and a multi-index $(n_1, \dots, n_m) \in \mathbb{N}^m$, the class

$$\mathcal{U}_m^{(n_1, \dots, n_m)} := \bigcup_{E_1, \dots, E_m, F} \mathcal{U}_m^{(n_1, \dots, n_m)}(^{n_1}E_1, \dots, ^{n_m}E_m; F)$$

is called an *ideal of (n_1, \dots, n_m) -homogeneous polynomials*.

Natural examples can be easily found in the context of absolutely summing operators (we follow the usual notation of the subject).

Definition 4.3. Let $m \in \mathbb{N}$, $p, q_1, \dots, q_m \geq 1$ and let E_1, \dots, E_m, F be Banach spaces. A continuous (n_1, \dots, n_m) -homogeneous polynomial $P : E_1 \times \dots \times E_m \rightarrow F$ is said to be *absolutely* $(p; q_1, \dots, q_m)$ -*summing* (or $(p; q_1, \dots, q_m)$ -*summing*) if $P \left(x_j^{(1)}, \dots, x_j^{(m)} \right)_{j=1}^\infty \in \ell_p(F)$ for all $\left(x_j^{(k)} \right)_{j=1}^\infty \in \ell_{q_k}^w(E_k)$, $k = 1, \dots, m$.

We shall denote by $\mathcal{P}_{as(p, q_1, \dots, q_m)}({}^{n_1}E_1, \dots, {}^{n_m}E_m; F)$ the subspace of all $P \in \mathcal{P}({}^{n_1}E_1, \dots, {}^{n_m}E_m; F)$ which are absolutely $(p; q_1, \dots, q_m)$ -summing.

Next we present a quite more demanding definition than the previous one.

Definition 4.4. Let $m \in \mathbb{N}$, $p, q_1, \dots, q_m \geq 1$ and let E_1, \dots, E_m, F be Banach spaces. A continuous (n_1, \dots, n_m) -homogeneous polynomial $P : E_1 \times \dots \times E_m \rightarrow F$ is said to be *multiple* $(p; q_1, \dots, q_m)$ -*summing* if $P \left(x_{j_1}^{(1)}, \dots, x_{j_m}^{(m)} \right)_{j_1, \dots, j_m=1}^\infty \in \ell_p(F)$ for all $\left(x_{j_k}^{(k)} \right)_{j_k=1}^\infty \in \ell_{q_k}^w(E_k)$, $k = 1, \dots, m$.

Having in mind the success of the multilinear theory of absolutely summing operators, the above notions are a natural topic for future investigation.

5. COHERENCE AND COMPATIBILITY

The extension of an operator ideal to polynomial and multilinear mappings is not always a trivial question. For example, the ideal of absolutely summing operators has at least eight possible extensions to higher degrees (see, for example, [15, 19, 23, 27, 28, 35, 36, 38] and references therein). The almost summing operators is another example of operator ideal which has several different possible extensions to the setting of multilinear and polynomial ideals [13, 32, 33]. Motivated by questions about finding a more adequate and less artificial extension of a given operator ideal being able to preserve its main properties and essence, several concepts like ideals closed for scalar multiplication and ideals closed under differentiation were first created in [10] (see also [14] for related notions). With the same aim of filtering good polynomial extensions of a given operator ideal and exclusively targeted to polynomial ideals, the notions of coherent sequence and compatible ideals were introduced by D. Carando et al. in [16] (see also in [17, 18]). In this section we are mainly interested in these two last concepts and we recall them after fixing the following notation:

- If $P \in \mathcal{P}({}^m E; F)$, then $\overset{\vee}{P}$ denotes the unique symmetric m -linear mapping associated to P .
- If $P \in \mathcal{P}({}^m E; F)$, then P_{a^k} is the $(m - k)$ -homogeneous polynomial in $\mathcal{P}({}^{m-k} E; F)$ defined by

$$P_{a^k}(x) := \overset{\vee}{P}(a, \dots, a, x, \dots, x).$$

Next we recall the definitions of coherent and compatible polynomial ideals. Our notation essentially follows [16].

Definition 5.1 (Compatible polynomial ideals [16]). Let \mathcal{I} be a normed ideal of linear operators. The normed ideal of n -homogeneous polynomials \mathcal{U}_n is *compatible* with \mathcal{I} if there exist positive constants α_1 and α_2 such that for all Banach spaces E and F , the following conditions hold:

(cp 1): For each $P \in \mathcal{U}_n({}^n E; F)$ and $a \in E$, the mapping $P_{a^{n-1}}$ belongs to $\mathcal{I}(E; F)$ and

$$\|P_{a^{n-1}}\|_{\mathcal{I}} \leq \alpha_1 \|P\|_{\mathcal{U}_n} \|a\|^{n-1}.$$

(cp 2): For each $P \in \mathcal{I}(E; F)$ and $\gamma \in E'$, the mapping $\gamma^{n-1}P$ belongs to $\mathcal{U}_n({}^n E; F)$ and

$$\|\gamma^{n-1}P\|_{\mathcal{U}_n} \leq \alpha_2 \|\gamma\|^{n-1} \|P\|_{\mathcal{I}}.$$

Definition 5.2 (Coherent polynomial ideals [16]). Consider a sequence $(\mathcal{U}_k)_{k=1}^N$, where for each k , \mathcal{U}_k is a normed ideal of k -homogeneous polynomials and N is eventually infinite. The sequence $(\mathcal{U}_k)_{k=1}^N$ is a *coherent sequence of polynomial ideals* if there exist positive constants β_1 and β_2 such that for all Banach spaces E and F , the following conditions hold for $k = 1, \dots, N - 1$:

(ch 1): For each $P \in \mathcal{U}_{k+1}({}^{k+1}E; F)$ and $a \in E$, the mapping P_a belongs to $\mathcal{U}_k({}^kE; F)$ and

$$\|P_a\|_{\mathcal{U}_k} \leq \beta_1 \|P\|_{\mathcal{U}_{k+1}} \|a\|.$$

(ch 2): For each $P \in \mathcal{U}_k({}^kE; F)$ and $\gamma \in E'$, the mapping γP belongs to $\mathcal{U}_{k+1}({}^{k+1}E; F)$ and

$$\|\gamma P\|_{\mathcal{U}_{k+1}} \leq \beta_2 \|\gamma\| \|P\|_{\mathcal{U}_k}.$$

The philosophy brought by the above concepts is concerning to being able to transit between different levels of homogeneity of a given polynomial ideal preserving the interconnection and the spirit of the original level ($n = 1$). Motivated by the fact that an operator ideal \mathcal{I} can be always extended (at least in an abstract sense) not only to polynomials but also to the multilinear settings (see [8]), D. Pellegrino and J. Ribeiro [34] proposed a significant new approach to coherence and compatibility which deals simultaneously with polynomials and multilinear operator ideals by considering pairs $(\mathcal{U}_k, \mathcal{M}_k)_{k=1}^\infty$, where \mathcal{U}_k is a (quasi-) normed ideal of k -homogeneous polynomials and \mathcal{M}_k is a (quasi-) normed ideal of k -linear mappings. We recall how it was done in the next definition (we essentially follow the notation in [34]).

Definition 5.3 (Compatible pair of ideals [34]). Let \mathcal{I} be a normed operator ideal and $N \in (\mathbb{N} \setminus \{1\}) \cup \{\infty\}$. A sequence $(\mathcal{U}_k, \mathcal{M}_k)_{k=1}^N$, with $\mathcal{U}_1 = \mathcal{M}_1 = \mathcal{I}$, is compatible with \mathcal{I} if there exist positive constants $\alpha_1, \alpha_2, \alpha_3$ such that for all Banach spaces E, E_1, \dots, E_n and F , the following conditions hold for all $n \in \{2, \dots, N\}$:

(cp-i): For each $k \in \{1, \dots, n\}$, $T \in \mathcal{M}_n(E_1, \dots, E_n; F)$ and $a_j \in E_j$ for all $j \in \{1, \dots, n\} \setminus \{k\}$, the mapping $T(a_1, \dots, a_{k-1}, \cdot, a_{k+1}, \dots, a_n)$ belongs to $\mathcal{U}(E_k; F)$ and

$$\begin{aligned} & \|T(a_1, \dots, a_{k-1}, \cdot, a_{k+1}, \dots, a_n)\|_{\mathcal{I}} \\ & \leq \alpha_1 \|T\|_{\mathcal{M}_n} \|a_1\| \cdots \|a_{k-1}\| \|a_{k+1}\| \cdots \|a_n\|. \end{aligned}$$

(cp-ii): For each $P \in \mathcal{U}_n({}^nE; F)$ and $a \in E$, the mapping $P_{a^{n-1}}$ belongs to $\mathcal{I}(E; F)$ and

$$\|P_{a^{n-1}}\|_{\mathcal{I}} \leq \alpha_2 \max \left\{ \left\| \bigvee P \right\|_{\mathcal{M}_n} \|P\|_{\mathcal{U}_n} \right\} \|a\|^{n-1}.$$

(cp-iii): For each $T \in \mathcal{I}(E_n; F)$ and $\gamma_j \in E'_j$ for $j = 1, \dots, n-1$, the mapping $\gamma_1 \cdots \gamma_{n-1} T$ belongs to $\mathcal{M}_n(E_1, \dots, E_n; F)$ and

$$\|\gamma_1 \cdots \gamma_{n-1} T\|_{\mathcal{M}_n} \leq \alpha_3 \|\gamma_1\| \cdots \|\gamma_{n-1}\| \|T\|_{\mathcal{I}}.$$

(cp-iv): For each $P \in \mathcal{I}(E; F)$ and $\gamma \in E'$, the mapping $\gamma^{n-1} P$ belongs to $\mathcal{U}_n({}^nE; F)$.

(cp-v): P belongs to $\mathcal{U}_n({}^nE; F)$ if, and only if, $\bigvee P$ belongs to $\mathcal{M}_n({}^nE; F)$.

Definition 5.4 (Coherent pair of ideals [34]). Let \mathcal{I} be a normed operator ideal and $N \in \mathbb{N} \cup \{\infty\}$. A sequence $(\mathcal{U}_k, \mathcal{M}_k)_{k=1}^N$, with $\mathcal{U}_1 = \mathcal{M}_1 = \mathcal{I}$, is coherent if there exist positive constants $\beta_1, \beta_2, \beta_3$ such that for all Banach spaces E, E_1, \dots, E_{k+1} and F , the following conditions hold for all $k = 1, \dots, N-1$:

(ch-i): For each $T \in \mathcal{M}_{k+1}(E_1, \dots, E_{k+1}; F)$ and $a_j \in E_j$ for $j = 1, \dots, k+1$, the mapping $T(\cdot, \dots, \cdot, a_j, \cdot, \dots, \cdot)$ belongs to $\mathcal{M}_k(E_1, \dots, E_{j-1}, \cdot, E_{j+1}, \dots, E_{k+1}; F)$ and

$$\|T(\cdot, \dots, \cdot, a_j, \cdot, \dots, \cdot)\|_{\mathcal{M}_k} \leq \beta_1 \|T\|_{\mathcal{M}_{k+1}} \|a_j\|.$$

(ch-ii): For each $P \in \mathcal{U}_{k+1}({}^{k+1}E; F)$ and $a \in E$, the mapping P_a belongs to $\mathcal{U}_k({}^kE; F)$ and

$$\|P_a\|_{\mathcal{U}_k} \leq \beta_2 \max \left\{ \left\| \bigvee P \right\|_{\mathcal{M}_{k+1}} \|P\|_{\mathcal{U}_{k+1}} \right\} \|a\|.$$

(ch-iii): For each $T \in \mathcal{M}_k(E_1, \dots, E_k; F)$ and $\gamma \in E'_{k+1}$, the mapping γT belongs to $\mathcal{M}_{k+1}(E_1, \dots, E_{k+1}; F)$ and

$$\|\gamma T\|_{\mathcal{M}_{k+1}} \leq \beta_3 \|\gamma\| \|T\|_{\mathcal{M}_k}.$$

(ch-iv): For each $P \in \mathcal{U}_k({}^kE; F)$ and $\gamma \in E'$, the mapping γP belongs to $\mathcal{U}_{k+1}({}^{k+1}E; F)$.

(ch-v): P belongs to $\mathcal{U}_k({}^k E; F)$ if, and only if, $\bigvee P$ belongs to $\mathcal{M}_k({}^k E; F)$.

The compatible and coherent pair of ideals concepts above was the first attempt in order to have an evaluating method to extending ideals addressed simultaneously to polynomials and multilinear mappings. Besides, this approach goes further than the previous one when achieves the coherence for the canonical sequence $(\mathcal{P}_k)_{k=1}^\infty$ composed by the ideals of continuous k -homogeneous polynomials with the sup norm, which is not coherent according to Definition 5.2 (this remark appears in [16] and it is based on estimates for the norms of certain homogeneous polynomials used in [14, Proposition 8.5]).

5.1. The extension to multipolynomial ideals. In this section we invoke the multipolynomials to present a more general definition of coherence and compatibility including the first one and, under some restrictions, the second one as particular cases. Since it recovers Definitions 5.1 and 5.2, one can automatically have in hands the desired unified treatment for the polynomial and multilinear ideals extensions. Indeed, it suffices to set $m = 1$ (and $n_1 = n \in \mathbb{N}$) in our forthcoming approach to get the Carando's polynomial cases and, furthermore, we can go even further to the multilinear cases by setting $m > 1$ and $n_1 = \dots = n_m = 1$. Pellegrino–Ribeiro's approach for pairs of ideals (5.3 and 5.4) is also naturally recovered by considering a multipolynomial ideals family which has such pair as its ends.

Definition 5.5 (Compatible multipolynomial ideals). Let \mathcal{I} be a normed operator ideal. The (quasi-) normed ideal of (n_1, \dots, n_m) -homogeneous polynomials $\mathcal{U}_m^{(n_1, \dots, n_m)}$ is *compatible* with \mathcal{I} if there exist positive constants α_1 and α_2 such that for all Banach spaces E_1, \dots, E_m , the following conditions hold:

(CP 1): For each $k \in \{1, \dots, m\}$, $P \in \mathcal{U}_m^{(n_1, \dots, n_m)}({}^{n_1} E_1, \dots, {}^{n_m} E_m; F)$, $a_k \in E_k$ and $x_j \in E_j$ for all $j \in \{1, \dots, m\} \setminus \{k\}$, the mapping $P_{a_k^{n_k-1}}(x_1, \dots, x_{k-1}, \cdot, x_{k+1}, \dots, x_m)$ belongs to $\mathcal{I}(E_k; F)$ and

$$\begin{aligned} & \left\| P_{a_k^{n_k-1}}(x_1, \dots, x_{k-1}, \cdot, x_{k+1}, \dots, x_m) \right\|_{\mathcal{I}} \\ & \leq \alpha_1 \|P\|_{\mathcal{U}_m^{(n_1, \dots, n_m)}} \|x_1\|^{n_1} \dots \|x_{k-1}\|^{n_{k-1}} \|a_k\|^{n_k-1} \|x_{k+1}\|^{n_{k+1}} \dots \|x_m\|^{n_m}. \end{aligned}$$

(CP 2): For each $P \in \mathcal{I}({}^{n_m} E_m; F)$ and $\gamma_j \in E'_j$ for $j = 1, \dots, m$, the mapping $\gamma_1^{n_1} \dots \gamma_{m-1}^{n_{m-1}} \gamma_m^{n_m-1} P$ belongs to $\mathcal{U}_m^{(n_1, \dots, n_m)}({}^{n_1} E_1, \dots, {}^{n_m} E_m; F)$ and

$$\left\| \gamma_1^{n_1} \dots \gamma_{m-1}^{n_{m-1}} \gamma_m^{n_m-1} P \right\|_{\mathcal{U}_m^{(n_1, \dots, n_m)}} \leq \alpha_2 \|\gamma_1\|^{n_1} \dots \|\gamma_{m-1}\|^{n_{m-1}} \|\gamma_m\|^{n_m-1} \|P\|_{\mathcal{I}}.$$

(CP 3): For each $P \in \mathcal{P}({}^{n_1} E_1, \dots, {}^{n_m} E_m; F)$, $k \in \{1, \dots, m\}$ and $x_j \in E_j$ for all $j \in \{1, \dots, m\} \setminus \{k\}$, $P(x_1, \dots, x_{k-1}, \cdot, x_{k+1}, \dots, x_m)$ belongs to $\mathcal{U}_1^{(n_k)}({}^{n_k} E_k; F)$ if, and only if, $P(x_1, \dots, x_{k-1}, \cdot, x_{k+1}, \dots, x_m)^\vee$ belongs to $\mathcal{U}_{n_k}^{(1, \dots, 1)}({}^1 E_k, \dots, {}^1 E_k; F)$.

Next we extend the coherence notion to multipolynomials.

Definition 5.6 (Coherent multipolynomial ideals). Let \mathcal{I} be a normed operator ideal. A family $\{\mathcal{U}_\alpha\}_{\alpha \in I}$, with $\mathcal{U}_1^{(1)} = \mathcal{I}$, is *coherent* if there exist positive constants $\beta_1, \beta_2, \beta_3, \beta_4$ such that for all Banach spaces E, E_1, \dots, E_{k+1} and F , the following conditions hold for all $k \in \mathbb{N}$ and for all multi-index $\alpha = (n_1, \dots, n_k) \in \mathbb{N}^k$.

(CH 1): For each $j \in \{1, \dots, k\}$, $P \in \mathcal{U}_k^{(n_1, \dots, n_{j+1}, \dots, n_k)}({}^{n_1} E_1, \dots, {}^{n_{j+1}} E_{j+1}, \dots, {}^{n_k} E_k; F)$, $a_j \in E_j$ and $x_i \in E_i$ for all $i \in \{1, \dots, k\} \setminus \{j\}$, the n_j -homogeneous polynomial $P_{a_j}(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_k)$ belongs to $\mathcal{U}_1^{(n_j)}({}^{n_j} E_j; F)$ and

$$\begin{aligned} & \left\| P_{a_j}(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_k) \right\|_{\mathcal{U}_1^{(n_j)}} \\ & \leq \beta_1 \|P\|_{\mathcal{U}_k^{(n_1, \dots, n_{j+1}, \dots, n_k)}} \|x_1\|^{n_1} \dots \|x_{j-1}\|^{n_{j-1}} \|a_j\| \|x_{j+1}\|^{n_{j+1}} \dots \|x_k\|^{n_k}. \end{aligned}$$

(CH 2): For each $P \in \mathcal{U}_{k+1}^{(n_1, \dots, n_{k+1})} ({}^{n_1}E_1, \dots, {}^{n_{k+1}}E_{k+1}; F)$ and $a_j \in E_j$ for $j = 1, \dots, k+1$, the multipolynomial $P(\cdot, \dots, \cdot, a_j, \cdot, \dots, \cdot)$ belongs to $\mathcal{U}_k^{(n_1, \dots, n_{j-1}, n_{j+1}, \dots, n_{k+1})} ({}^{n_1}E_1, \dots, {}^{n_{j-1}}E_{j-1}, {}^{n_{j+1}}E_{j+1}, \dots, {}^{n_{k+1}}E_{k+1}; F)$ and

$$\|P(\cdot, \dots, \cdot, a_j, \cdot, \dots, \cdot)\|_{\mathcal{U}_k^{(n_1, \dots, n_{j-1}, n_{j+1}, \dots, n_{k+1})}} \leq \beta_2 \|P\|_{\mathcal{U}_{k+1}^{(n_1, \dots, n_{k+1})}} \|a_j\|^{n_j}.$$

(CH 3): For each $j \in \{1, \dots, k\}$, $P \in \mathcal{U}_k^{(n_1, \dots, n_j, \dots, n_k)} ({}^{n_1}E_1, \dots, {}^{n_j}E_j, \dots, {}^{n_k}E_k; F)$ and $\gamma_j \in E'_j$, the mapping $\gamma_j P$ belongs to $\mathcal{U}_k^{(n_1, \dots, n_j+1, \dots, n_k)} ({}^{n_1}E_1, \dots, {}^{n_j+1}E_j, \dots, {}^{n_k}E_k; F)$ and

$$\|\gamma_j P\|_{\mathcal{U}_k^{(n_1, \dots, n_j+1, \dots, n_k)}} \leq \beta_3 \|\gamma_j\| \|P\|_{\mathcal{U}_k^{(n_1, \dots, n_j, \dots, n_k)}}.$$

(CH 4): For each $P \in \mathcal{U}_k^{(n_1, \dots, n_k)} ({}^{n_1}E_1, \dots, {}^{n_k}E_k; F)$ and $\gamma \in E'_{k+1}$, the mapping γP belongs to $\mathcal{U}_{k+1}^{(n_1, \dots, n_k, 1)} ({}^{n_1}E_1, \dots, {}^{n_k}E_k, {}^1E_{k+1}; F)$ and

$$\|\gamma P\|_{\mathcal{U}_{k+1}^{(n_1, \dots, n_k, 1)}} \leq \beta_4 \|\gamma\| \|P\|_{\mathcal{U}_k^{(n_1, \dots, n_k)}}.$$

(CH 5): For each $P \in \mathcal{P} ({}^{n_1}E_1, \dots, {}^{n_k}E_k; F)$, $j \in \{1, \dots, k\}$ and $x_i \in E_i$ for all $i \in \{1, \dots, k\} \setminus \{j\}$, $P(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_k)$ belongs to $\mathcal{U}_1^{(n_j)} ({}^{n_j}E_j; F)$ if, and only if, $P(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_k)^\vee$ belongs to $\mathcal{U}_{n_j}^{(1, \dots, 1)} ({}^1E_j, \dots, {}^1E_j; F)$.

Remark 5.7. As it was previously said, if we set $m = 1$ ($n_1 = n$ and $E_1 = E_m = E$) then our Definition 5.5 recovers Definition 5.1 and goes even further to the multilinear cases when $m > 1$ and $n_1 = \dots = n_m = 1$. If we fix $k = 1$ and makes n_1 vary in $\{1, \dots, N-1\}$ then the items (CH 1) and (CH 3) of our Definition 5.6 recover Definition 5.2.

It is worth noting that sequences of pair of ideals are just particular cases of multipolynomial ideals families and therefore, in this sense, one could recover the compatible (or coherent) pair of ideals just by considering a family $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ of multipolynomial ideals. Let's take a look at a way to extract compatible (or coherent) pair of ideals from such a given family.

Remark 5.8. From now on we shall denote by $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ the multipolynomial ideals family such that $\mathcal{U}_\alpha := \mathcal{U}_m^{(n_1, \dots, n_m)}$ is a (quasi-) normed ideal of α -homogeneous polynomials, for each multi-index $\alpha = (n_1, \dots, n_m) \in I = \bigcup_{m=1}^{\infty} \mathbb{N}^m$.

Definition 5.9. We shall say that a family of multipolynomial ideals $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ is compatible with a normed operator ideal \mathcal{I} when \mathcal{U}_α is compatible with \mathcal{I} for all multi-index $\alpha \in I$.

Proposition 5.10. Let $N \in (\mathbb{N} \setminus \{1\}) \cup \{\infty\}$. If a family $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ is compatible with the operator ideal $\mathcal{I} := \mathcal{U}_1^{(1)}$ (resp. coherent), then the sequence of pairs of ideals $\left(\mathcal{U}_1^{(n)}, \mathcal{U}_n^{(1, \dots, 1)}\right)_{n=1}^N$ is compatible with \mathcal{I} (resp. coherent).

Proof. We treat the compatible case (the coherent case is analogous). Let E, E_1, \dots, E_n and F be Banach spaces for a fixed (but arbitrary) $n \in \{2, \dots, N\}$. Applying the hypothesis with $\alpha = (1, \dots, 1) \in \mathbb{N}^n$ then (cp-i) follows from (CP 1) with $x_j = a_j$ and any $a_k \in E_k$, and (cp-iii) follows from (CP 2) with any $\gamma_m \in E_m$. Applying the hypothesis with $\alpha = n$ then (cp-ii) follows from (CP 1) with $E_k = E_1 = E$ and $a_k = a_1 = a$, (cp-iv) follows from (CP 2) with $E_m = E_1 = E$ and $\gamma_m = \gamma_1 = \gamma$ and, finally, (cp-v) follows from (CP 3). \square

Corollary 5.11. For a given $N \in (\mathbb{N} \setminus \{1\}) \cup \{\infty\}$, let $(\mathcal{Q}_k)_{k=1}^N$ be a sequence of (quasi-) normed ideals of k -homogeneous polynomials and let $(\mathcal{M}_k)_{k=1}^N$ be a sequence of (quasi-) normed ideals of k -linear mappings. Suppose that $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ is a family of multipolynomial ideals such that

$$\left(\mathcal{U}_1^{(k)}\right)_{k=1}^N = (\mathcal{Q}_k)_{k=1}^N \text{ and } \left(\mathcal{U}_k^{(1, \dots, 1)}\right)_{k=1}^N = (\mathcal{M}_k)_{k=1}^N.$$

Then the compatibility of the family $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ with $\mathcal{I} = \mathcal{U}_1^{(1)}$ (resp. the coherence) implies the compatibility of the sequence $(\mathcal{Q}_k, \mathcal{M}_k)_{k=1}^N$ with \mathcal{I} (resp. the coherence).

6. MULTIPOLYNOMIAL HYPER-IDEALS

Recently in papers [11] and [12] the authors introduced and developed the respective notions of hyper-ideals of multilinear operators and homogeneous polynomials between Banach spaces. While the well studied notions of ideals of multilinear operators (multi-ideals) as well as polynomial ideals relies on the composition with linear operators (the so-called ideal property), the notion proposed by the authors, called now as hyper-ideal property, considers in [11] the compositions with multilinear operators and, under the polynomial viewpoint, considers in [12] the compositions with homogeneous polynomials. Historically speaking, the hyper-ideal property has already been studied individually for some specific classes, see, e.g. [22, 41, 42], and then [11, 12] started the systematic study of the classes satisfying this stronger condition. The aim of this section is to invoke the multipolynomials again, as we have done before, to generalize and propose a unified approach for all these isolated notions of hyper-ideals of operators (multilinear and polynomial) which have been studied separately so far.

Definition 6.1 (Hyper-ideal of multilinear operators [11]). A *hyper-ideal of multilinear operators* is a subclass \mathcal{H} of the class of all continuous multilinear operators between Banach spaces such that for all $n \in \mathbb{N}$ and all Banach spaces E_1, \dots, E_n and F , the components

$$\mathcal{H}(E_1, \dots, E_n; F) := \mathcal{L}(E_1, \dots, E_n; F) \cap \mathcal{H}$$

satisfy:

- (ha): $\mathcal{H}(E_1, \dots, E_n; F)$ is a linear subspace of $\mathcal{L}(E_1, \dots, E_n; F)$ which contains the n -linear operators of finite type ;
- (hb): The **hyper-ideal property**: given natural numbers n and $1 \leq m_1 < \dots < m_n$ and Banach spaces $G_1, \dots, G_{m_n}, E_1, \dots, E_n, F$ and H , if $B_1 \in \mathcal{L}(G_1, \dots, G_{m_1}; E_1), \dots, B_n \in \mathcal{L}(G_{m_{n-1}+1}, \dots, G_{m_n}; E_n), A \in \mathcal{H}(E_1, \dots, E_n; F)$ and $t \in \mathcal{L}(F; H)$, then
$$t \circ A \circ (B_1, \dots, B_n) \in \mathcal{H}(G_1, \dots, G_{m_n}; H).$$

Moreover, \mathcal{H} is said to be a (*quasi*-) *normed hyper-ideal of multilinear operators* if there exists a map $\|\cdot\|_{\mathcal{H}} : \mathcal{H} \rightarrow [0, \infty)$ satisfying:

- (h1): $\|\cdot\|_{\mathcal{H}}$ restricted to $\mathcal{H}(E_1, \dots, E_n; F)$ is a (*quasi*-) norm, for all $n \in \mathbb{N}$ and all Banach spaces E_1, \dots, E_n, F ;
- (h2): $\|I_n : \mathbb{K}^n \rightarrow \mathbb{K}, I_n(\lambda_1, \dots, \lambda_n) = \lambda_1 \cdots \lambda_n\|_{\mathcal{H}} = 1$, for all $n \in \mathbb{N}$;
- (h3): The **hyper-ideal inequality**: if $B_1 \in \mathcal{L}(G_1, \dots, G_{m_1}; E_1), \dots, B_n \in \mathcal{L}(G_{m_{n-1}+1}, \dots, G_{m_n}; E_n), A \in \mathcal{H}(E_1, \dots, E_n; F)$ and $t \in \mathcal{L}(F; H)$, then
$$\|t \circ A \circ (B_1, \dots, B_n)\|_{\mathcal{H}} \leq \|t\| \|A\|_{\mathcal{H}} \|B_1\| \cdots \|B_n\|.$$

When all the components $\mathcal{H}(E_1, \dots, E_n; F)$ are complete under the (*quasi*-) norm $\|\cdot\|_{\mathcal{H}}$ above, then $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is called a (*quasi*-) *Banach hyper-ideal of multilinear operators*.

It is plain that every (normed, quasi-normed, Banach, quasi-Banach) hyper-ideal is a (normed, quasi-normed, Banach, quasi-Banach) multi-ideal.

Definition 6.2 (Polynomial hyper-ideal [12]). A *polynomial hyper-ideal* is a subclass \mathcal{Q} of the class of all continuous homogeneous polynomials between Banach spaces such that for all $n \in \mathbb{N}$ and all Banach spaces E and F , the components

$$\mathcal{Q}(^n E; F) := \mathcal{P}(^n E; F) \cap \mathcal{Q}$$

satisfy:

- (pa): $\mathcal{Q}(^n E; F)$ is a linear subspace of $\mathcal{P}(^n E; F)$ which contains the n -homogeneous polynomials of finite type;
- (pb): The **hyper-ideal property**: given $m, n \in \mathbb{N}$ and Banach spaces E, F, G and H , if $Q \in \mathcal{P}(^m G; E), P \in \mathcal{Q}(^n E; F)$ and $t \in \mathcal{L}(F; H)$, then
$$t \circ P \circ Q \in \mathcal{Q}(^{mn} G; H).$$

If there exist a map $\|\cdot\|_{\mathcal{Q}} : \mathcal{Q} \rightarrow [0, \infty)$ and a sequence $(C_j)_{j=1}^{\infty}$ of real numbers with $C_j \geq 1$ for every $j \in \mathbb{N}$ and $C_1 = 1$, such that:

- (p1): $\|\cdot\|_{\mathcal{Q}}$ restricted to $\mathcal{Q}(^n E; F)$ is a (quasi-) norm, for all $n \in \mathbb{N}$ and all Banach spaces E and F ;
- (p2): $\|I_n : \mathbb{K} \rightarrow \mathbb{K}, I_n(\lambda) = \lambda^n\|_{\mathcal{Q}} = 1$, for all $n \in \mathbb{N}$;
- (p3): The **hyper-ideal inequality**: if $Q \in \mathcal{P}(^m G; E)$, $P \in \mathcal{Q}(^n E; F)$ and $t \in \mathcal{L}(F; H)$, then

$$\|t \circ P \circ Q\|_{\mathcal{Q}} \leq (C_m)^n \|t\| \|P\|_{\mathcal{Q}} \|Q\|^n,$$

then $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ is called a *(quasi-) normed polynomial $(C_j)_{j=1}^{\infty}$ -hyper-ideal*. When all the components $\mathcal{Q}(^n E; F)$ are complete under the (quasi-) norm $\|\cdot\|_{\mathcal{Q}}$ above, then $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ is called a *(quasi-) Banach polynomial $(C_j)_{j=1}^{\infty}$ -hyper-ideal*.

When $C_j = 1$ for every $j \in \mathbb{N}$, we simply say that \mathcal{Q} is a *(quasi-) normed/(quasi-) Banach polynomial hyper-ideal*. When the hyper-ideal property (and inequality) holds for every $n \in \mathbb{N}$, but only for $m = 1$, we say that \mathcal{Q} is a *(quasi-) normed/(quasi-) Banach polynomial ideal* (remember that $C_1 = 1$).

Definition 6.3 (Polynomial two-sided ideal [12]). A *polynomial two-sided ideal* is a subclass \mathcal{Q} of the class of all continuous homogeneous polynomials between Banach spaces such that for all $n \in \mathbb{N}$ and all Banach spaces E and F , the components

$$\mathcal{Q}(^n E; F) := \mathcal{P}(^n E; F) \cap \mathcal{Q}$$

satisfy:

- (ts-a): $\mathcal{Q}(^n E; F)$ is a linear subspace of $\mathcal{P}(^n E; F)$ which contains the n -homogeneous polynomials of finite type;
- (ts-b): The **two-sided ideal property**: given $m, n, r \in \mathbb{N}$ and Banach spaces E, F, G and H , if $Q \in \mathcal{P}(^m G; E)$, $P \in \mathcal{Q}(^n E; F)$ and $R \in \mathcal{P}(^r F; H)$, then

$$R \circ P \circ Q \in \mathcal{Q}(^{mnr} G; H).$$

If there exist a map $\|\cdot\|_{\mathcal{Q}} : \mathcal{Q} \rightarrow [0, \infty)$ and a sequence $(C_j, K_j)_{j=1}^{\infty}$ of pairs of real numbers with $C_j, K_j \geq 1$ for every $j \in \mathbb{N}$ and $C_1 = K_1 = 1$, such that:

- (ts-1): $\|\cdot\|_{\mathcal{Q}}$ restricted to $\mathcal{Q}(^n E; F)$ is a (quasi-) norm, for all $n \in \mathbb{N}$ and all Banach spaces E and F ;
- (ts-2): $\|I_n : \mathbb{K} \rightarrow \mathbb{K}, I_n(\lambda) = \lambda^n\|_{\mathcal{Q}} = 1$, for all $n \in \mathbb{N}$;
- (ts-3): The **two-sided ideal inequality**: if $Q \in \mathcal{P}(^m G; E)$, $P \in \mathcal{Q}(^n E; F)$ and $R \in \mathcal{P}(^r F; H)$, then

$$\|R \circ P \circ Q\|_{\mathcal{Q}} \leq K_r (C_m)^{rn} \|R\| \|P\|_{\mathcal{Q}} \|Q\|^{rn},$$

then $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ is called a *(quasi-) normed polynomial $(C_j, K_j)_{j=1}^{\infty}$ -two-sided ideal*. When all the components $\mathcal{Q}(^n E; F)$ are complete under the (quasi-) norm $\|\cdot\|_{\mathcal{Q}}$ above, then $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ is called a *(quasi-) Banach polynomial $(C_j, K_j)_{j=1}^{\infty}$ -two-sided ideal*.

When $C_j = K_j = 1$ for every $j \in \mathbb{N}$, we simply say that \mathcal{Q} is a *(quasi-) normed/(quasi-) Banach polynomial two-sided ideal*.

Remark 6.4. The condition $C_1 = K_1 = 1$ guarantees that every (normed, quasi-normed, Banach, quasi-Banach) polynomial $(C_j, K_j)_{j=1}^{\infty}$ -two-sided ideal is a (normed, quasi-normed, Banach, quasi-Banach) polynomial $(C_j)_{j=1}^{\infty}$ -hyper-ideal; and that, as we mentioned before, every (normed, quasi-normed, Banach, quasi-Banach) polynomial $(C_j)_{j=1}^{\infty}$ -hyper-ideal is a (normed, quasi-normed, Banach, quasi-Banach) polynomial ideal.

Next we extend the above notions to the multipolynomials.

Definition 6.5 (Multipolynomial hyper-ideal). A *hyper-ideal of multipolynomials* (or *multipolynomial hyper-ideals*) is a subclass \mathcal{H} of the class of all continuous multipolynomials between Banach spaces

such that for all $n \in \mathbb{N}$, multi-index $(k_1, \dots, k_n) \in \mathbb{N}^n$ and all Banach spaces E_1, \dots, E_n and F , the components

$$\mathcal{H}_n^{(k_1, \dots, k_n)}(k_1 E_1, \dots, k_n E_n; F) := \mathcal{P}(k_1 E_1, \dots, k_n E_n; F) \cap \mathcal{H}$$

satisfy:

- (Ha): $\mathcal{H}_n^{(k_1, \dots, k_n)}(k_1 E_1, \dots, k_n E_n; F)$ is a linear subspace of $\mathcal{P}(k_1 E_1, \dots, k_n E_n; F)$ which contains the (k_1, \dots, k_n) -homogeneous polynomials of finite type ;
- (Hb): The **hyper-ideal property**: given natural numbers n , $1 \leq m_1 < \dots < m_n$, r_1, \dots, r_{m_n} , k_1, \dots, k_n and r and Banach spaces $G_1, \dots, G_{m_n}, E_1, \dots, E_n, F$ and H , if $Q_1 \in \mathcal{P}(r_1 G_1, \dots, r_{m_1} G_{m_1}; E_1), \dots, Q_n \in \mathcal{P}(r_{m_{n-1}+1} G_{m_{n-1}+1}, \dots, r_{m_n} G_{m_n}; E_n)$, $P \in \mathcal{H}_n^{(k_1, \dots, k_n)}(k_1 E_1, \dots, k_n E_n; F)$ and $R \in \mathcal{P}(r F; H)$, then

$$R \circ P \circ (Q_1, \dots, Q_n) \in \mathcal{H}_{m_n}^{(r_1 k_1 r, \dots, r_{m_1} k_1 r, \dots, r_{m_n} k_n r)}(r_1 k_1 r G_1, \dots, r_{m_n} k_n r G_{m_n}; H).$$

If there exist a map $\|\cdot\|_{\mathcal{H}} : \mathcal{H} \rightarrow [0, \infty)$ and a sequence $(C_j, K_j)_{j=1}^{\infty}$ of pairs of real numbers with $C_j, K_j \geq 1$ for every $j \in \mathbb{N}$ and $C_1 = K_1 = 1$, such that:

- (H1): $\|\cdot\|_{\mathcal{H}}$ restricted to $\mathcal{H}_n^{(k_1, \dots, k_n)}(k_1 E_1, \dots, k_n E_n; F)$ is a (quasi-) norm, for all $n \in \mathbb{N}$, multi-index $(k_1, \dots, k_n) \in \mathbb{N}^n$ and all Banach spaces E_1, \dots, E_n, F ;
- (H2): $\|I_n^{(k_1, \dots, k_n)} : \mathbb{K}^n \rightarrow \mathbb{K}, I_n^{(k_1, \dots, k_n)}(\lambda_1, \dots, \lambda_n) = \lambda_1^{k_1} \dots \lambda_n^{k_n}\|_{\mathcal{H}} = 1$, for all $n \in \mathbb{N}$ and $(k_1, \dots, k_n) \in \mathbb{N}^n$;
- (H3): The **hyper-ideal inequality**: if $Q_1 \in \mathcal{P}(r_1 G_1, \dots, r_{m_1} G_{m_1}; E_1), \dots, Q_n \in \mathcal{P}(r_{m_{n-1}+1} G_{m_{n-1}+1}, \dots, r_{m_n} G_{m_n}; E_n)$, $P \in \mathcal{H}_n^{(k_1, \dots, k_n)}(k_1 E_1, \dots, k_n E_n; F)$ and $R \in \mathcal{P}(r F; H)$, then

$$\begin{aligned} & \|R \circ P \circ (Q_1, \dots, Q_n)\|_{\mathcal{H}} \\ & \leq K_r (C_{r_1} \dots C_{r_{m_1}})^{r k_1} \dots (C_{r_{m_{n-1}+1}} \dots C_{r_{m_n}})^{r k_n} \|R\| \|P\|_{\mathcal{H}}^r \|Q_1\|^{r k_1} \dots \|Q_n\|^{r k_n}, \end{aligned}$$

then $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is called a *(quasi-) normed multipolynomial $(C_j, K_j)_{j=1}^{\infty}$ -hyper-ideal*. When all the components $\mathcal{H}_n^{(k_1, \dots, k_n)}(k_1 E_1, \dots, k_n E_n; F)$ are complete under the (quasi-) norm $\|\cdot\|_{\mathcal{H}}$ above, then $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is called a *(quasi-) Banach multipolynomial $(C_j, K_j)_{j=1}^{\infty}$ -hyper-ideal*.

When $C_j = K_j = 1$ for every $j \in \mathbb{N}$, we simply say that \mathcal{H} is a *(quasi-) normed/(quasi-) Banach multipolynomial hyper-ideal*.

Remark 6.6. Our multipolynomial hyper-ideals definition is more general and unifying in the sense that it recovers the multilinear and polynomial cases. Indeed, setting $n = 1 = m_1$ we get Definition 6.3 if $r > 1$ and Definition 6.2 if $r = 1$. In the other end, setting $k_1 = \dots = k_n = r_1 = \dots = r_{m_n} = r = 1$ we recover Definition 6.1. Finally, it is plain that every (normed, quasi-normed, Banach, quasi-Banach) multipolynomial hyper-ideal is a (normed, quasi-normed, Banach, quasi-Banach) multipolynomial ideal.

7. A BOHNENBLUST–HILLE INEQUALITY FOR MULTIPOLYNOMIALS

In this section we present a unified version to the Bohnenblust–Hille inequalities [6] for homogeneous polynomials and for multilinear forms. The theory of Bohnenblust–Hille inequalities has been exhaustively investigated in recent years (see, for instance [2, 3, 20, 21, 25, 30, 31, 37, 43], and the references therein).

Let $\alpha = (\alpha_j)_{j=1}^{\infty}$ be a sequence in $\mathbb{N} \cup \{0\}$ and, as usual, define $|\alpha| = \sum_{j=1}^{\infty} \alpha_j$; in this case we also denote $\mathbf{x}^{\alpha} := \prod_j x_j^{\alpha_j}$. An m -homogeneous polynomial $P : c_0 \rightarrow \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}) is denoted by

$$P(x) = \sum_{|\alpha|=m} c_{\alpha}(P) \mathbf{x}^{\alpha}.$$

We recall that the norm of P is given by $\|P\| := \sup_{\|x\| \leq 1} |P(x)|$.

The Bohnenblust–Hille inequality for homogeneous polynomials [6] asserts that

Theorem 7.1 (Polynomial Bohnenblust–Hille inequality). *Let m be a positive fixed integer. The following assertions are equivalent:*

(i): *There exists a constant $C_{\mathbb{K},m} \geq 1$ such that*

$$\left(\sum_{|\alpha|=m} |c_\alpha(P)|^p \right)^{\frac{1}{p}} \leq C_{\mathbb{K},m} \|P\|$$

for all continuous m -homogeneous polynomial $P : c_0 \rightarrow \mathbb{K}$;

(ii):

$$p \geq \frac{2m}{m+1}.$$

We also have the Bohnenblust–Hille inequality for multilinear forms [6]:

Theorem 7.2 (Multilinear Bohnenblust–Hille inequality). *Let m be a positive fixed integer. The following assertions are equivalent:*

(i): *There exists a constant $C_{\mathbb{K},m} \geq 1$ such that*

$$\left(\sum_{i_1, \dots, i_m=1}^{\infty} |T(e_{i_1}, \dots, e_{i_m})|^p \right)^{\frac{1}{p}} \leq C_{\mathbb{K},m} \|T\|$$

for all continuous m -linear forms $T : c_0 \times \dots \times c_0 \rightarrow \mathbb{K}$;

(ii):

$$p \geq \frac{2m}{m+1}.$$

Our next step is to unify these results above. We shall observe that a (n_1, \dots, n_m) -homogeneous polynomial $P : c_0 \times \dots \times c_0 \rightarrow \mathbb{K}$ can be always written as

$$P(x^{(1)}, \dots, x^{(m)}) = \sum_{|\alpha^{(1)}|=n_1, \dots, |\alpha^{(m)}|=n_m} c_{\alpha^{(1)} \dots \alpha^{(m)}}(P) (x^{(1)})^{\alpha^{(1)}} \dots (x^{(m)})^{\alpha^{(m)}}$$

where, as we have previously defined, $(\alpha_j^{(i)})_{j=1}^{\infty}$ is a sequence in $\mathbb{N} \cup \{0\}$, $|\alpha^{(i)}| = \sum_{j=1}^{\infty} \alpha_j^{(i)}$ and

$$(x^{(i)})^{\alpha^{(i)}} = \prod_j (x_j^{(i)})^{\alpha_j^{(i)}}, \text{ for } i = 1, \dots, m.$$

Next we invoke the notion of multipolynomials to unify Theorems 7.1 and 7.2. In fact, these are respectively the particular cases $m = 1$ and $n_1 = \dots = n_m = 1$ of the following theorem:

Theorem 7.3 (Multipolynomial Bohnenblust–Hille inequality). *Let n_1, \dots, n_m and m be fixed positive integers. The following assertions are equivalent:*

(i): *There is a constant $C_{n_1, \dots, n_m} \geq 1$ such that*

$$\left(\sum_{|\alpha^{(1)}|=n_1, \dots, |\alpha^{(m)}|=n_m} |c_{\alpha^{(1)} \dots \alpha^{(m)}}(P)|^p \right)^{\frac{1}{p}} \leq C_{n_1, \dots, n_m} \|P\|$$

for all (n_1, \dots, n_m) -homogeneous polynomial $P : c_0 \times \dots \times c_0 \rightarrow \mathbb{K}$.

(ii):

$$p \geq \frac{2 \left(\sum_{j=1}^m n_j \right)}{\left(\sum_{j=1}^m n_j \right) + 1}.$$

Proof. (ii) \Rightarrow (i): It suffices to prove the assertion for

$$p_0 = \frac{2 \left(\sum_{j=1}^m n_j \right)}{\left(\sum_{j=1}^m n_j \right) + 1}.$$

Let $Q : c_0 \rightarrow \mathbb{K}$ be the $(n_1 + \dots + n_m)$ -homogeneous polynomial given by

$$Q(z) := P \left((z_j)_{j \in \mathbb{N}_1}, \dots, (z_j)_{j \in \mathbb{N}_m} \right),$$

where $\mathbb{N} = \mathbb{N}_1 \cup \dots \cup \mathbb{N}_m$ is a disjoint union with $\text{card}(\mathbb{N}_j) = \text{card}(\mathbb{N})$, for $j = 1, \dots, m$. Note that since we are dealing with the sup norm we have

$$\|Q\| \leq \|P\|$$

and

$$\sum_{|\beta|=n_1+\dots+n_m} |c_\beta(Q)|^p = \sum_{|\alpha^{(1)}|=n_1, \dots, |\alpha^{(m)}|=n_m} |c_{\alpha^{(1)} \dots \alpha^{(m)}}(P)|^p,$$

for all p . By the Polynomial Bohnenblust-Hille Inequality there exists a constant $C_{n_1+\dots+n_m} \geq 1$ such that

$$\begin{aligned} \left(\sum_{|\alpha^{(1)}|=n_1, \dots, |\alpha^{(m)}|=n_m} |c_{\alpha^{(1)} \dots \alpha^{(m)}}(P)|^{p_0} \right)^{\frac{1}{p_0}} &= \left(\sum_{|\beta|=n_1+\dots+n_m} |c_\beta(Q)|^{p_0} \right)^{\frac{1}{p_0}} \\ &\leq C_{n_1+\dots+n_m} \|Q\| \\ &= C_{n_1+\dots+n_m} \|P\|. \end{aligned}$$

(i) \Rightarrow (ii): Let

$$\begin{aligned} T_r &: c_0 \times \dots \times c_0 \rightarrow \mathbb{K} \\ T_r(x^{(1)}, \dots, x^{(M)}) &= \sum_{i_1, \dots, i_M=1}^r \pm x_{i_1}^{(1)} \dots x_{i_M}^{(M)} \end{aligned}$$

be the M -linear operator given by the Kahane-Salem-Zygmund inequality (see, [2, Lemma 6.1]) with

$$M = \sum_{j=1}^m n_j.$$

Define $P_r : \overbrace{c_0 \times \dots \times c_0}^m \rightarrow \mathbb{K}$ by

$$P_r(x^{(1)}, \dots, x^{(m)}) = T_r \left(\underbrace{\left(x_j^{(1)} \right)_{j \in \mathbb{N}_1^{(1)}}, \dots, \left(x_j^{(1)} \right)_{j \in \mathbb{N}_{n_1}^{(1)}}}_{n_1}, \dots, \underbrace{\left(x_j^{(m)} \right)_{j \in \mathbb{N}_1^{(m)}}, \dots, \left(x_j^{(m)} \right)_{j \in \mathbb{N}_{n_m}^{(m)}}}_{n_m} \right),$$

where

$$\begin{aligned}\mathbb{N} &= \mathbb{N}_1^{(1)} \cup \dots \cup \mathbb{N}_{n_1}^{(1)} \\ &\vdots \\ \mathbb{N} &= \mathbb{N}_1^{(m)} \cup \dots \cup \mathbb{N}_{n_m}^{(m)}\end{aligned}$$

are disjoint unions with $\text{card}(\mathbb{N}_k^{(i)}) = \text{card}(\mathbb{N})$, for $i = 1, \dots, m$ and $k = 1, \dots, n_i$. Note that P_r is an (n_1, \dots, n_m) -homogeneous polynomial and $\|P_r\| \leq \|T_r\|$. Moreover,

$$\begin{aligned}\sum_{|\alpha|=n_1+\dots+n_m} |c_\alpha(P_r)|^p &= \sum_{i_1, \dots, i_M=1}^{\infty} |T_r(e_{i_1}, \dots, e_{i_M})|^p \\ &= r^M\end{aligned}$$

for all p . Since

$$\|T_r\| \leq K_M r^{\frac{M+1}{2}},$$

by (i) we conclude that

$$(r^M)^{\frac{1}{p}} \leq C_{n_1, \dots, n_m} K_M r^{\frac{M+1}{2}}$$

for all positive integers r . Thus

$$\frac{M}{p} \leq \frac{M+1}{2}$$

and the proof is done. \square

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